1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f.
x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph beside it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph above it.

3. (a) See the Evaluation Theorem at the beginning of Section 5.3.

(b) See the Net Change Theorem after Example 6 in Section 5.3.

4. $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Evaluation Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous.

7. See the Fundamental Theorem of Calculus after Example 5 in Section 5.4.

8. (a) See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

(b) See Formula 5.6.1 or 5.6.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.

9. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 5.9. We would expect the best estimate to be given by Simpson's Rule.

10. See Definitions 1(a), (b), and (c) in Section 5.10.

11. See Definitions 3(b), (a), and (c) in Section 5.10.

12. See the Comparison Theorem after Example 8 in Section 5.10.

13. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it in Section 5.4.

1. True by Property 2 of the Integral in Section 5.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 5.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$\int_0^1 x\, f(x)\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$ (a constant) while $x \int_0^1 1\, dx = x\,[x]_0^1 = x \cdot 1 = x$ (a variable).

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\, dx = \int_0^1 x\, dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\, dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True by the Net Change Theorem.

7. True by Comparison Property 7 of the Integral in Section 5.2.

8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.6(b).

10. True.

$$\int_{-5}^{5} \left(ax^2 + bx + c\right) dx = \int_{-5}^{5} \left(ax^2 + c\right) dx + \int_{-5}^{5} bx\, dx$$
$$= 2\int_0^5 \left(ax^2 + c\right) dx \quad \text{[by 5.5.6(a)]} \quad + 0 \quad \text{[by 5.5.6(b)]}$$

11. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2 - 1}\, dx = \int_0^1 \frac{x}{x^2 - 1}\, dx + \int_1^4 \frac{x}{x^2 - 1}\, dx \text{ and}$$
$$\int_0^1 \frac{x}{x^2 - 1}\, dx = \lim_{t \to 1^-} \int_0^t \frac{x}{x^2 - 1}\, dx = \lim_{t \to 1^-} \left[\frac{1}{2}\ln\left|x^2 - 1\right|\right]_0^t = \lim_{t \to 1^-} \frac{1}{2}\ln\left|t^2 - 1\right| = \infty$$

So the integral diverges.

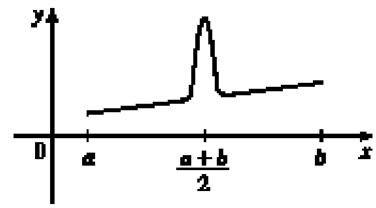
12. True by Theorem 5.10.2 with $p = \sqrt{2} > 1$.

13. False. See the remarks and Figure 4 before Example 1 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \leq 2$.

14. True by FTC1.

15. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

16. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



17. False. See Exercise 51 in Section 5.10.

18. True. If f is continuous on $[0, \infty)$, then $\int_0^1 f(x)\,dx$ is finite. Since $\int_1^\infty f(x)\,dx$ is finite, so is $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$.

19. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

20. True.

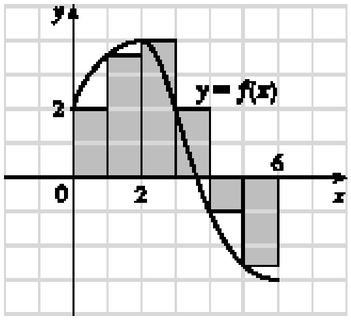
$$\int_a^\infty [f(x) + g(x)]\,dx = \lim_{t\to\infty} \int_a^t [f(x) + g(x)]\,dx = \lim_{t\to\infty} \left(\int_a^t f(x)\,dx + \int_a^t g(x)\,dx\right)$$
$$= \lim_{t\to\infty} \int_a^t f(x)\,dx + \lim_{t\to\infty} \int_a^t g(x)\,dx \quad \left[\text{since both limits in the sum exist}\right]$$
$$= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

Since the two integrals are finite, so is their sum.

21. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x) + g(x)]\,dx = 0$ [convergent].

22. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

1. (a)



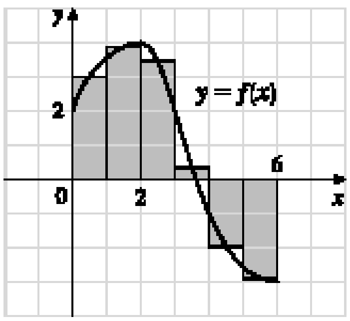
$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$

$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$

$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

(b)



$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$

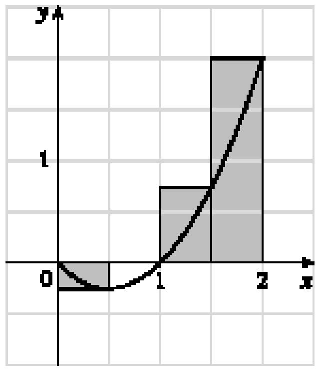
$$= f(\overline{x}_1)\cdot 1 + f(\overline{x}_2)\cdot 1 + f(\overline{x}_3)\cdot 1 + f(\overline{x}_4)\cdot 1 + f(\overline{x}_5)\cdot 1 + f(\overline{x}_6)\cdot 1$$

$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$

$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \quad \Rightarrow$

$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$
$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

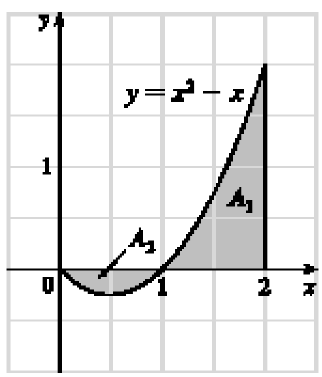
The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the area of the rectangle below the x-axis. (The second rectangle vanishes.)

(b) $\int_0^2 (x^2 - x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\ \Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$
$$= \lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} - 2\cdot\frac{n+1}{n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 2\left(1+\frac{1}{n}\right)\right] = \frac{4}{3}\cdot 1\cdot 2 - 2\cdot 1 = \frac{2}{3}$$

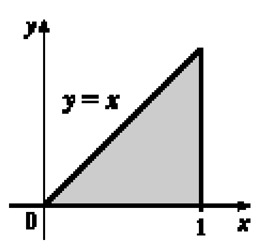
(c) $\int_0^2 (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$

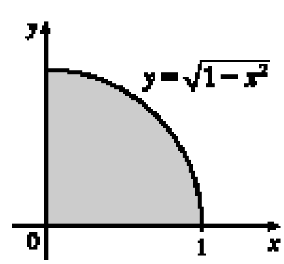
(d)



$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1-x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx = I_1 + I_2.$





I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

4. On $[0,\pi]$, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sin x_i\,\Delta x=\int_0^{\pi}\sin x\,dx=\left[-\cos x\right]_0^{\pi}=-(-1)-(-1)=2.$

5. $\int_0^6 f(x)\,dx=\int_0^4 f(x)\,dx+\int_4^6 f(x)\,dx \quad\Rightarrow\quad 10=7+\int_4^6 f(x)\,dx \quad\Rightarrow\quad \int_4^6 f(x)\,dx=10-7=3$

6. (a) $f(x)=e^{3x}$, $\Delta x=(2-0)/n=2/n$, and $x_i=2i/n \quad\Rightarrow$

$$\int_0^2 e^{3x}\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}e^{3(2i/n)}\left(\frac{2}{n}\right)$$

$$\overset{\text{CAS}}{=}\lim_{n\to\infty}\frac{2e^{6/n}\left(e^6-1\right)}{n(e^{6/n}-1)}\overset{\text{CAS}}{=}\frac{e^6-1}{3}\approx 134.14$$

(b) $\int_0^2 e^{3x}\,dx=\left[\frac{1}{3}e^{3x}\right]_0^2=\frac{1}{3}\left(e^6-1\right)$, as in part (a).

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt=0$, and $c(0)\neq 0$. Now notice that $b>0$ when c is increasing, and that $c>0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By the Evaluation Theorem (FTC2), $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right)dx=\left[e^{\arctan x}\right]_0^1=e^{\pi/4}-1$

(b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\arctan x}\,dx=0$ since this is the derivative of a constant.

(c) By FTC1, $\displaystyle\frac{d}{dx}\int_0^x e^{\arctan t}\,dt=e^{\arctan x}$.

9. $\int_1^2\left(8x^3+3x^2\right)dx=\left[8\cdot\frac{1}{4}x^4+3\cdot\frac{1}{3}x^3\right]_1^2=\left[2x^4+x^3\right]_1^2=\left(2\cdot 2^4+2^3\right)-(2+1)=40-3=37$

10. $\displaystyle\int_0^T\left(x^4-8x+7\right)dx=\left[\frac{1}{5}x^5-4x^2+7x\right]_0^T=\left(\frac{1}{5}T^5-4T^2+7T\right)-0=\frac{1}{5}T^5-4T^2+7T$

11. $\int_0^1\left(1-x^9\right)dx=\left[x-\frac{1}{10}x^{10}\right]_0^1=\left(1-\frac{1}{10}\right)-0=\frac{9}{10}$

12. Let $u=1-x$, so $du=-dx$ and $dx=-du$. When $x=0$, $u=1$; when $x=1$, $u=0$. Thus,
$\int_0^1(1-x)^9\,dx=\int_1^0 u^9(-du)=\int_0^1 u^9\,du=\frac{1}{10}\left[u^{10}\right]_0^1=\frac{1}{10}(1-0)=\frac{1}{10}$.

13. $\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du = \int_1^9 (u^{-1/2}-2u)\,du = \left[2u^{1/2}-u^2\right]_1^9 = (6-81)-(2-1) = -76$

14. $\int_0^1 \left(\sqrt[4]{u}+1\right)^2\,du = \int_0^1 (u^{1/2}+2u^{1/4}+1)\,du = \left[\frac{2}{3}u^{3/2}+\frac{8}{5}u^{5/4}+u\right]_0^1 = \left(\frac{2}{3}+\frac{8}{5}+1\right)-0 = \frac{49}{15}$

15. $u = x^2+1$, $du = 2x\,dx$, so $\int_0^1 \frac{x}{x^2+1}\,dx = \int_1^2 \frac{1}{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\ln u\right]_1^2 = \frac{1}{2}\ln 2.$

16. $\int_0^1 \frac{1}{x^2+1}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1-\tan^{-1}0 = \frac{\pi}{4}-0 = \frac{\pi}{4}$

17. Let $u = v^3$, so $du = 3v^2\,dv$. When $v=0$, $u=0$; when $v=1$, $u=1$. Thus,
$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1-0) = \frac{1}{3}\sin 1.$

18. Let $u = 3\pi t$, so $du = 3\pi\,dt$. When $t=0$, $u=1$; when $t=1$, $u=3\pi$. Thus,
$\int_0^1 \sin(3\pi t)\,dt = \int_0^{3\pi} \sin u\left(\frac{1}{3\pi}\,du\right) = \frac{1}{3\pi}\left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1-1) = \frac{2}{3\pi}.$

19. $\int_0^1 e^{\pi t}\,dt = \left[\frac{1}{\pi}e^{\pi t}\right]_0^1 = \frac{1}{\pi}(e^{\pi}-1)$

20. Let $u = 2-3x$, so $du = -3\,dx$. When $x=1$, $u=-1$; when $x=2$, $u=-4$. Thus,
$\int_1^2 \frac{1}{2-3x}\,dx = \int_{-1}^{-4} \frac{1}{u}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\left[\ln|u|\right]_{-1}^{-4} = -\frac{1}{3}(\ln 4-\ln 1) = -\frac{1}{3}\ln 4.$

21. Let $u = x^2+4x$. Then $du = (2x+4)\,dx = 2(x+2)\,dx$, so
$\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int u^{-1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot 2u^{1/2}+C = \sqrt{u}+C = \sqrt{x^2+4x}+C.$

22. Integrate by parts with $u = \ln x$, $dv = x^3\,dx \quad\Rightarrow\quad du = dx/x$, $v = x^4/4$:
$\int_1^2 x^3\ln x\,dx = \left[\frac{1}{4}x^4\ln x\right]_1^2 - \frac{1}{4}\int_1^2 x^3\,dx = 4\ln 2-\frac{1}{16}\left[x^4\right]_1^2 = 4\ln 2-\frac{15}{16}.$

23. $\int_0^5 \frac{x}{x+10}\,dx = \int_0^5 \left(1 - \frac{10}{x+10}\right) dx = \Big[x - 10\ln(x+10)\Big]_0^5$
$= 5 - 10\ln 15 + 10\ln 10 = 5 + 10\ln\frac{10}{15} = 5 + 10\ln\frac{2}{3}$

24. $\int_0^5 ye^{-0.6y}\,dy \quad \begin{bmatrix} u = y, & dv = e^{-0.6y}\,dy, \\ du = dy & v = -\frac{5}{3}e^{-0.6y} \end{bmatrix} = \left[-\frac{5}{3}ye^{-0.6y}\right]_0^5 - \int_0^5 \left(-\frac{5}{3}e^{-0.6y}\right) dy$
$= -\frac{25}{3}e^{-3} - \frac{25}{9}\left[e^{-0.6y}\right]_0^5 = -\frac{25}{3}e^{-3} - \frac{25}{9}\left(e^{-3} - 1\right)$
$= -\frac{25}{3}e^{-3} - \frac{25}{9}e^{-3} + \frac{25}{9} = \frac{25}{9} - \frac{100}{9}e^{-3}$

25. $\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta = \Big[\ln(1+\sin\theta)\Big]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$

26. $\int_1^4 \frac{dt}{(2t+1)^3} \quad \begin{bmatrix} u = 2t+1, \\ du = 2\,dt \end{bmatrix} = \int_3^9 \frac{\frac{1}{2}\,du}{u^3} = \frac{-1}{4}\left[\frac{1}{u^2}\right]_3^9 = -\frac{1}{4}\left(\frac{1}{81} - \frac{1}{9}\right) = -\frac{1}{4}\left(-\frac{8}{81}\right) = \frac{2}{81}$

27. $\int_1^4 x^{3/2}\ln x\,dx \quad \begin{bmatrix} u = \ln x, & dv = x^{3/2}\,dx, \\ du = dx/x & v = \frac{2}{5}x^{5/2} \end{bmatrix} = \frac{2}{5}\Big[x^{5/2}\ln x\Big]_1^4 - \frac{2}{5}\int_1^4 x^{3/2}\,dx$
$= \frac{2}{5}(32\ln 4 - \ln 1) - \frac{2}{5}\left[\frac{2}{5}x^{5/2}\right]_1^4$
$= \frac{2}{5}(64\ln 2) - \frac{4}{25}(32 - 1)$
$= \frac{128}{5}\ln 2 - \frac{124}{25} \quad \left(\text{or } \frac{64}{5}\ln 4 - \frac{124}{25}\right)$

28. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\int \sin x\,\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

29. $\frac{1}{t^2+6t+8} = \frac{1}{(t+2)(t+4)} = \frac{A}{t+2} + \frac{B}{t+4}$. Multiply both sides by $(t+2)(t+4)$ to get $1 = A(t+4) + B(t+2)$. Substituting -4 for t gives $1 = -2B \quad \Leftrightarrow \quad B = -\frac{1}{2}$. Substituting -2 for t gives $1 = 2A \quad \Leftrightarrow \quad A = \frac{1}{2}$. Thus,
$\int \frac{dt}{t^2+6t+8} = \int \left(\frac{1/2}{t+2} - \frac{1/2}{t+4}\right) dt = \frac{1}{2}\ln|t+2| - \frac{1}{2}\ln|t+4| + C = \frac{1}{2}\ln\left|\frac{t+2}{t+4}\right| + C.$

30. Let $u = x^2$. Then $du = 2x\,dx$, so $\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}\left(x^2\right) + C$.

31. Let $w = \sqrt[3]{x}$. Then $w^3 = x$ and $3w^2\,dw = dx$, so $\int e^{\sqrt[3]{x}}\,dx = \int e^w \cdot 3w^2\,dw = 3I$. To evaluate I, let $u = w^2$, $dv = e^w\,dw \Rightarrow du = 2w\,dw$, $v = e^w$, so $I = \int w^2 e^w\,dw = w^2 e^w - \int 2we^w\,dw$. Now let $U = w$, $dV = e^w\,dw \Rightarrow dU = dw$, $V = e^w$. Thus, $I = w^2e^w - 2\left[we^w - \int e^w\,dw\right] = w^2e^w - 2we^w + 2e^w + C_1$, and hence $3I = 3e^w(w^2 - 2w + 2) + C = 3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$.

32. Let $u = \tan^{-1} x$, $dv = dx \Rightarrow du = \dfrac{1}{1+x^2}\,dx$, $v = x$:

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \tfrac{1}{2}\ln\left(1+x^2\right) + C.$$

33. $I = \displaystyle\int_0^\infty \frac{\ln x}{x^4}\,dx = \int_0^1 \frac{\ln x}{x^4}\,dx + \int_1^\infty \frac{\ln x}{x^4}\,dx = I_1 + I_2$. Integrate by parts with $u = \ln x$, $dv = dx/x^4 \Rightarrow du = dx/x$, $v = -1/\left(3x^3\right)$:

$$\int \frac{\ln x}{x^4}\,dx = -\frac{\ln x}{3x^3} + \frac{1}{3}\int \frac{1}{x^4}\,dx = -\frac{\ln x}{3x^3} + \frac{1}{3}\left(-\frac{1}{3x^3}\right) + C = -\frac{1}{9}\cdot\frac{3\ln x + 1}{x^3} + C$$

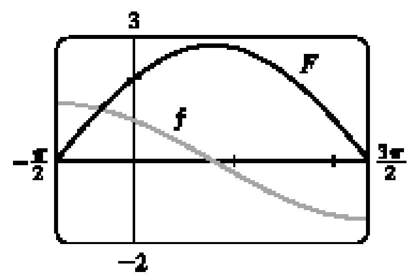
$$I_1 = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x^4}\,dx = -\frac{1}{9}\lim_{t\to 0^+}\left[\frac{3\ln x + 1}{x^3}\right]_t^1 = -\frac{1}{9}\lim_{t\to 0^+}\left[1 - \frac{3\ln t + 1}{t^3}\right] = -\infty$$

So I_1 diverges and hence, I diverges. Divergent

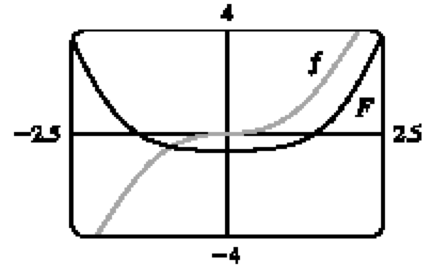
34. $\displaystyle\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx = 0$ by Theorem 5.5.6(b), since $f(x) = \dfrac{\sin x}{1+x^2}$ is an odd function.

35. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$

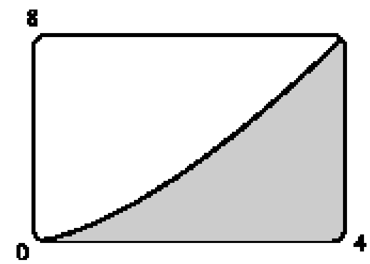


36. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so



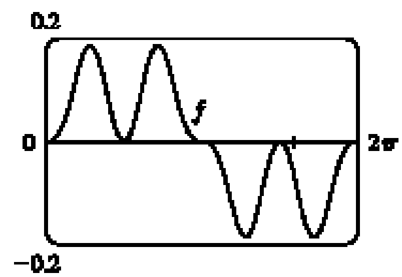
$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{1/2}\left[\left(x^2+1\right) - 3\right] + C \\ &= \tfrac{1}{3}\sqrt{x^2+1}\left(x^2-2\right) + C\end{aligned}$$

37. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate



$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{2}{5}(4)^{5/2} = \frac{64}{5} = 12.8.$

38. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x\,\sin^3 x\,dx$ is equal to 0. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\left(1 - \cos^2 x\right)\sin x\,dx$ and let $u = \cos x \quad\Rightarrow$ $du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2\left(1 - u^2\right)(-du) = 0$.



39. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4}\,dt \quad\Rightarrow\quad F'(x) = \sqrt{1+x^4}$.

40. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$\begin{aligned}\frac{d}{dx}\int_1^{\cos x} \sqrt[3]{1-t^2}\,dt &= \frac{d}{du}\int_1^{u} \sqrt[3]{1-t^2}\,dt \cdot \frac{du}{dx} = \sqrt[3]{1-u^2}\,(-\sin x) \\ &= \sqrt[3]{1-\cos^2 x}\,(-\sin x) = -\sin x\,\sqrt[3]{\sin^2 x} = -(\sin x)^{5/3}\end{aligned}$$

41. $y = \int_{\sqrt{x}}^{x} \frac{e^t}{t}\,dt = \int_{\sqrt{x}}^{1} \frac{e^t}{t}\,dt + \int_{1}^{x} \frac{e^t}{t}\,dt = -\int_{1}^{\sqrt{x}} \frac{e^t}{t}\,dt + \int_{1}^{x} \frac{e^t}{t}\,dt \quad \Rightarrow$

$\frac{dy}{dx} = -\frac{d}{dx}\left(\int_{1}^{\sqrt{x}} \frac{e^t}{t}\,dt\right) + \frac{d}{dx}\left(\int_{1}^{x} \frac{e^t}{t}\,dt\right)$. Let $u = \sqrt{x}$. Then

$$\frac{d}{dx}\int_{1}^{\sqrt{x}} \frac{e^t}{t}\,dt = \frac{d}{dx}\int_{1}^{u} \frac{e^t}{t}\,dt = \frac{d}{du}\left(\int_{1}^{u} \frac{e^t}{t}\,dt\right)\frac{du}{dx} = \frac{e^u}{u}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x},$$

so $\frac{dy}{dx} = -\frac{e^{\sqrt{x}}}{2x} + \frac{e^x}{x}$.

42. $y = \int_{2x}^{3x+1} \sin(t^4)dt = \int_{2x}^{0} \sin(t^4)\,dt + \int_{0}^{3x+1} \sin(t^4)\,dt = \int_{0}^{3x+1} \sin(t^4)\,dt - \int_{0}^{2x} \sin(t^4)\,dt \quad \Rightarrow$

$y' = \sin\left[(3x+1)^4\right]\cdot\frac{d}{dx}(3x+1) - \sin\left[(2x)^4\right]\cdot\frac{d}{dx}(2x) = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$

43. $u = e^x \quad \Rightarrow \quad du = e^x\,dx$, so

$$\int e^x\sqrt{1-e^{2x}}\,dx = \int\sqrt{1-u^2}\,du \overset{30}{=} \tfrac{1}{2}u\sqrt{1-u^2} + \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\left[e^x\sqrt{1-e^{2x}} + \sin^{-1}(e^x)\right] + C$$

44. $\int \csc^5 t\,dt \overset{78}{=} -\frac{1}{4}\cot t\,\csc^3 t + \frac{3}{4}\int\csc^3 t\,dt$

$$\overset{72}{=} -\tfrac{1}{4}\cot t\,\csc^3 t + \tfrac{3}{4}\left[-\tfrac{1}{2}\csc t\,\cot t + \tfrac{1}{2}\ln|\csc t - \cot t|\right] + C$$
$$= -\tfrac{1}{4}\cot t\,\csc^3 t - \tfrac{3}{8}\csc t\,\cot t + \tfrac{3}{8}\ln|\csc t - \cot t| + C$$

45. $\int\sqrt{x^2+x+1}\,dx = \int\sqrt{x^2+x+\frac{1}{4}+\frac{3}{4}}\,dx = \int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

$$= \int\sqrt{u^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}\,du \quad \left[u = x+\tfrac{1}{2},\, du = dx\right]$$
$$\overset{21}{=} \tfrac{1}{2}u\sqrt{u^2+\tfrac{3}{4}} + \tfrac{3}{8}\ln\left(u+\sqrt{u^2+\tfrac{3}{4}}\right) + C$$
$$= \frac{2x+1}{4}\sqrt{x^2+x+1} + \tfrac{3}{8}\ln\left(x+\tfrac{1}{2}+\sqrt{x^2+x+1}\right) + C$$

46. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int\frac{\cot x\,dx}{\sqrt{1+2\sin x}} = \int\frac{du}{u\sqrt{1+2u}} \overset{\substack{57\text{ with}\\ a=1,\,b=2}}{=} \ln\left|\frac{\sqrt{1+2u}-1}{\sqrt{1+2u}+1}\right| + C = \ln\left|\frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1}\right| + C$$

47. $f(x) = \sqrt{1+x^4}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{10} = \dfrac{1}{10}$.

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2\,[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.090608$

(b) $M_{10} = \frac{1}{10}\left[f\left(\frac{1}{20}\right) + f\left(\frac{3}{20}\right) + f\left(\frac{5}{20}\right) + \cdots + f\left(\frac{19}{20}\right)\right] \approx 1.088840$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 4f(0.9) + f(1)] \approx 1.089429$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

48. $f(x) = \sqrt{\sin x}$, $\Delta x = \dfrac{\frac{\pi}{2} - 0}{10} = \dfrac{\pi}{20}$.

(a) $T_{10} = \frac{\pi}{20 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{2\pi}{20}\right) + \cdots + f\left(\frac{9\pi}{20}\right)\right] + f\left(\frac{\pi}{2}\right)\right\} \approx 1.185197$

(b) $M_{10} = \frac{\pi}{20}\left[f\left(\frac{\pi}{40}\right) + f\left(\frac{3\pi}{40}\right) + f\left(\frac{5\pi}{40}\right) + \cdots + f\left(\frac{17\pi}{40}\right) + f\left(\frac{19\pi}{40}\right)\right] \approx 1.201932$

(c) $S_{10} = \frac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.193089$

f is concave downward, so the Trapezoidal Rule gives us an underestimate, the Midpoint Rule gives an overestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

49. $f(x) = \left(1+x^4\right)^{1/2}$, $f'(x) = \frac{1}{2}\left(1+x^4\right)^{-1/2}\left(4x^3\right) = 2x^3\left(1+x^4\right)^{-1/2}$, $f''(x) = \left(2x^6+6x^2\right)\left(1+x^4\right)^{-3/2}$.

A graph of f'' on $[0,1]$ shows that it has its maximum at $x=1$, so $|f''(x)| \le f''(1) = \sqrt{8}$ on $[0,1]$. By taking $K = \sqrt{8}$, we find that the error in Exercise 47(a) is bounded by $\dfrac{K(b-a)^3}{12n^2} = \dfrac{\sqrt{8}}{1200} \approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024) = 0.0012$.

Note: Another way to estimate K is to let $x=1$ in the factor $2x^6+6x^2$ (maximizing the numerator) and let $x=0$ in the factor $\left(1+x^4\right)^{-3/2}$ (minimizing the denominator). Doing so gives us $K=8$ and errors of $0.00\overline{6}$ and $0.00\overline{3}$.

Using $K=8$ for the Trapezoidal Rule, we have $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \quad \Leftrightarrow \quad \dfrac{8(1-0)^3}{12n^2} \le \dfrac{1}{100{,}000} \quad \Leftrightarrow$

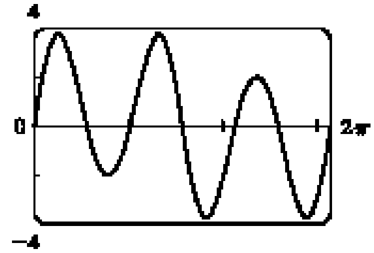
$n^2 \ge \dfrac{800{,}000}{12} \quad \Leftrightarrow \quad n \gtrsim 258.2$, so we should take $n = 259$.

For the Midpoint Rule, $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \quad \Leftrightarrow \quad n^2 \ge \dfrac{800{,}000}{24} \quad \Leftrightarrow \quad n \gtrsim 182.6$, so we should take $n = 183$.

50. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

51. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$f^{(4)}(x) = \sin(\sin x)\left[\cos^4 x + 7\cos^2 x - 3\right] + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]$$



From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.

(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x)\,dx \approx \tfrac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 5.9.4 with $K = 3.8$, and estimate the error as $|E_S| \leq \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \leq \frac{3.8\pi^5}{180n^4} \leq 0.00001$, so $n^4 \geq \frac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \quad \Rightarrow \quad n \geq 28.35$. Since n must be even for Simpson's Rule, we must have $n \geq 30$ to ensure the desired accuracy.

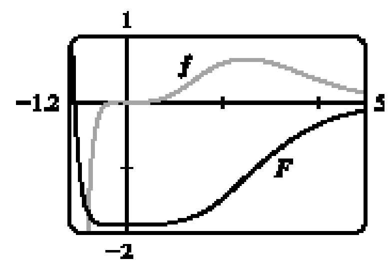
52. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx =$

$-\frac{1}{8}e^{-2x}\left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$

(d)



53. If $1 \leq x \leq 3$, then $\sqrt{1^2+3} \leq \sqrt{x^2+3} \leq \sqrt{3^2+3} \quad \Rightarrow \quad 2 \leq \sqrt{x^2+3} \leq 2\sqrt{3}$, so $2(3-1) \leq \int_1^3 \sqrt{x^2+3}\,dx \leq 2\sqrt{3}(3-1)$; that is, $4 \leq \int_1^3 \sqrt{x^2+3}\,dx \leq 4\sqrt{3}$.

54. On $[0,1]$, $x^4 \geq x^4 \cos x$ (since $0 \leq \cos x \leq 1$), so by Property 7, $\int_0^1 x^4\,dx \geq \int_0^1 x^4 \cos x\,dx$. Also, $x^4 \cos x \geq 0$, so by Property 6, $\int_0^1 x^4 \cos x\,dx \geq 0$. But $\int_0^1 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5} = 0.2$, so $0 \leq \int_0^1 x^4 \cos x\,dx \leq 0.2$.

55. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{2}(2x+1)^{-3}\,2\,dx$

$$= \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2}-\frac{1}{9}\right] = -\frac{1}{4}\left(0-\frac{1}{9}\right) = \frac{1}{36}$$

56. $I = \int_0^\infty \frac{\ln x}{x^4}\,dx = \int_0^1 \frac{\ln x}{x^4}\,dx + \int_1^\infty \frac{\ln x}{x^4}\,dx = I_1 + I_2$. Integrate by parts with $u = \ln x$, $dv = dx/x^4$ $\Rightarrow$ $du = dx/x$, $v = -1/\left(3x^3\right)$:

$$\int \frac{\ln x}{x^4}\,dx = -\frac{\ln x}{3x^3} + \frac{1}{3}\int \frac{1}{x^4}\,dx = -\frac{\ln x}{3x^3} + \frac{1}{3}\left(-\frac{1}{3x^3}\right) + C = -\frac{1}{9}\cdot\frac{3\ln x + 1}{x^3} + C$$

$$I_1 = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x^4}\,dx = -\frac{1}{9}\lim_{t\to 0^+}\left[\frac{3\ln x+1}{x^3}\right]_t^1 = -\frac{1}{9}\lim_{t\to 0^+}\left[1 - \frac{3\ln t + 1}{t^3}\right] = -\infty$$

So I_1 diverges and hence, I diverges. Divergent

57. $\int_{-\infty}^0 e^{-2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{-2x}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{2}e^{-2x}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}e^{-2t}\right) = \infty$. Divergent

58. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3}\frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\left[-\frac{1}{3}\ln|2-3x|\right]_0^t$$
$$= -\frac{1}{3}\lim_{t\to(2/3)^-}\Big[\ln|2-3t| - \ln 2\Big] = \infty$$

Since $\int_0^{2/3}\frac{1}{2-3x}\,dx$ diverges, so does $\int_0^1 \frac{1}{2-3x}\,dx$.

59. Let $u = \ln x$. Then $du = dx/x$, so $\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\ln x} + C$.

Thus, $\int_1^e \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to1^+}\int_t^e \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to1^+}\left[2\sqrt{\ln x}\right]_t^e = \lim_{t\to1^+}\left(2\sqrt{\ln e} - 2\sqrt{\ln t}\right) = 2\cdot 1 - 2\cdot 0 = 2.$

60. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int (u^2+2)\,du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus, $\displaystyle\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$

$$= \lim_{t\to 2^+}\left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}.$$

61. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1,\infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (5.10.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

62. $\displaystyle I = \int_0^\infty e^{ax}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99 \text{ with}\\ b=1}}{=} \lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)\right]_0^t$

$$= \lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t + \sin t) - \frac{1}{a^2+1}(a)\right] = \frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t) - a\right].$$

For $a \ge 0$, the limit does not exist due to oscillation. For $a < 0$, $\displaystyle\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t + \sin t)\right| \le e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

63. (a) displacement $= \int_0^5 (t^2 - t)\,dt = \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2\right]_0^5 = \tfrac{125}{3} - \tfrac{25}{2} = \tfrac{175}{6} = 29.1\overline{6}$ meters

(b) distance traveled $= \int_0^5 \left|t^2 - t\right|dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t - t^2)\,dt + \int_1^5 (t^2 - t)\,dt$

$$= \left[\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3\right]_0^1 + \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2\right]_1^5$$

$$= \tfrac{1}{2} - \tfrac{1}{3} - 0 + \left(\tfrac{125}{3} - \tfrac{25}{2}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right) = \tfrac{177}{6} = 29.5 \text{ meters}$$

64. $\Delta t = \left(\tfrac{10}{60} - 0\right)/\,10 = \tfrac{1}{60}$.

Distance traveled $= \int_0^{10} v\,dt \approx S_{10}$

$$= \tfrac{1}{60\cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]$$

$$= \tfrac{1}{180}(1544) = 8.5\overline{7} \text{ mi}$$

65. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem, $\int_0^3 r(t)\,dt = b(3) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003.

66. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population} &= \int_0^{24} r(t)\,dt \approx S_6 \\ &= \tfrac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)] \\ &= \tfrac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0] \\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067 \text{ bees}\end{aligned}$$

67. Both numerator and denominator approach 0 as $a \to 0$, so we use l'Hospital's Rule. (Note that we are differentiating *with respect to* a, since that is the quantity which is changing.) We also use FTC1:

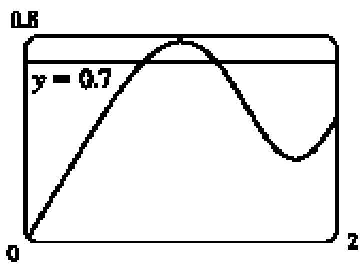
$$\lim_{a\to 0} T(x,t) = \lim_{a\to 0} \frac{C\int_0^a e^{-(x-u)^2/(4kt)}\,du}{a\sqrt{4\pi kt}} \overset{\text{H}}{=} \lim_{a\to 0} \frac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

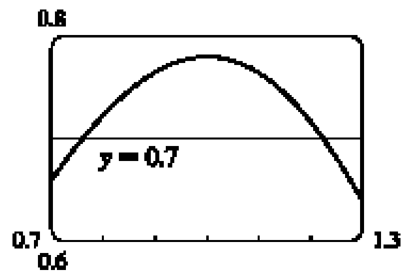
68. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus,

$C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi \quad \Leftrightarrow \quad 0 \le |x| \le 1$ or $\sqrt{4n-1} < |x| < \sqrt{4n+1}$, n any positive integer. So C is increasing on the intervals $[-1,1]$, $\left[\sqrt{3},\sqrt{5}\right]$, $\left[-\sqrt{5},-\sqrt{3}\right]$, $\left[\sqrt{7},3\right]$, $\left[-3,-\sqrt{7}\right]$, ...

(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right) \quad \Rightarrow$ $C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x \sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where $(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\quad \Leftrightarrow \quad \sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer. Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $\left(-\sqrt{2},0\right)$, $\left(\sqrt{2},2\right)$, $\left(-\sqrt{6},-2\right)$, $\left(\sqrt{6},2\sqrt{2}\right)$, ...

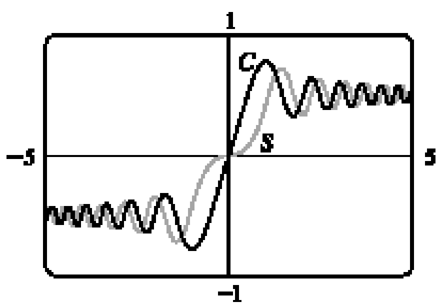
(c)





From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.4.4 and Exercise 5.4.23 for a discussion of $S(x)$.

69. Using FTC1, we differentiate both sides of the given equation, $\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t}f(t)\,dt$, and get

$f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x) \quad \Rightarrow \quad f(x)\left(1-e^{-x}\right) = e^{2x} + 2xe^{2x} \quad \Rightarrow \quad f(x) = \dfrac{e^{2x}(1+2x)}{1-e^{-x}}$.

70. $2\int_a^x f(t)\,dt = 2\sin x - 1 \quad \Rightarrow \quad \int_a^x f(t)\,dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$. We put $x = a$ into the last equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

71. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

72. Integrate by parts with $u = (\ln x)^n$, $dv = dx \quad \Rightarrow \quad du = n(\ln x)^{n-1} \cdot \frac{1}{x}\, dx$, $v = x$:
$\int (\ln x)^n\, dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$. Thus,

$$\int_0^1 (\ln x)^n\, dx = \lim_{t\to 0^+} \int_t^1 (\ln x)^n\, dx = \lim_{t\to 0^+} [x(\ln x)^n]_t^1 - n \lim_{t\to 0^+} \int_t^1 (\ln x)^{n-1}\, dx$$

$$= -\lim_{t\to 0^+} \frac{(\ln t)^n}{1/t} - n \int_0^1 (\ln x)^{n-1}\, dx = -n \int_0^1 (\ln x)^{n-1}\, dx,$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1 (\ln x)^n\, dx = (-1)^n n!$ for every positive integer n. For $n = 1$, we have $\int_0^1 (\ln x)^1\, dx = (-1) \int_0^1 (\ln x)^0\, dx = -\int_0^1 dx = -1$ (or $\int_0^1 \ln x\, dx = \lim_{t\to 0^+} [x \ln x - x]_t^1 = -1$). Assuming that the formula holds for n, we find that
$\int_0^1 (\ln x)^{n+1}\, dx = -(n+1) \int_0^1 (\ln x)^n\, dx = -(n+1)(-1)^n n! = (-1)^{n+1}(n+1)!$.
This is the formula for $n + 1$. Thus, the formula holds for all positive integers n by induction.

73. By the Fundamental Theorem of Calculus,
$\int_0^\infty f'(x)\, dx = \lim_{t\to\infty} \int_0^t f'(x)\, dx = \lim_{t\to\infty} [f(t) - f(0)] = \lim_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0)$.

74. The area $A(t) = \int_0^t \sin(x^2)\, dx$, and the area $B(t) = \frac{1}{2} t \sin(t^2)$. Since $\lim_{t\to 0^+} A(t) = 0 = \lim_{t\to 0^+} B(t)$, we can use l'Hospital's Rule:

$$\lim_{t\to 0^+} \frac{A(t)}{B(t)} \overset{\text{H}}{=} \lim_{t\to 0^+} \frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + \frac{1}{2} t[2t\cos(t^2)]} \qquad \text{[by FTC1 and the Product Rule]}$$

$$\overset{\text{H}}{=} \lim_{t\to 0^+} \frac{2t\cos(t^2)}{t\cos(t^2) - 2t^3\sin(t^2) + 2t\cos(t^2)} = \lim_{t\to 0^+} \frac{2\cos(t^2)}{3\cos(t^2) - 2t^2\sin(t^2)} = \frac{2}{3 - 0} = \frac{2}{3}$$